Novel Equivalent Circuit for $Z_{in}$ of an Arbitrarily Terminated Transmission Line

Timothy J. Maloney
Intel Corporation

Abstract: A new equivalent circuit for $Z_{in}$ of an arbitrarily terminated transmission line is presented, one which employs only lumped impedances and short and open line stubs in series-parallel combination. The new picture readily illustrates concepts of line matching and impedance inversion at quarter-wave frequencies, and also offers insights into transient and off-resonance response.

Many books on high-frequency electrical engineering ([1, 2] are just two) solve the one-dimensional wave equation for a transmission line and give the impedance looking into a transmission line terminated by load $Z_L$ as

$$Z_{in} = Z_0 \left[ \frac{Z_L \cosh(\gamma l) + Z_0 \sinh(\gamma l)}{Z_0 \cosh(\gamma l) + Z_L \sinh(\gamma l)} \right]. \quad (1)$$

For the customary RLGC transmission line of length $l$, $Z_0$ is the characteristic impedance $Z_0 = \sqrt{\frac{R + Ls}{G + Cs}}$ and the propagation constant is $\gamma = \sqrt{(R + Ls)(G + Cs)}$; both are in terms of complex frequency $s = \sigma + j\omega$ along with circuit elements $R$, $L$, $G$, and $C$. Figure 1 shows the termination of the transmission line by load $Z_L$; our interest is in the one-port characteristic impedance $Z_{in}$. Very often, $Z_{in}$ (or a succession of $Z_{in}$ solutions using successive $Z_L$ values) is sufficient for solving a problem because we want to know the effect of the load on the source. While there are 2-port pi and T models of transmission line sections that have some relation here [3], they will not be discussed.

Generations of students have stepped through the derivation of (1) and then have wrestled with its implications, dutifully using (1) to show impedance effects at even and odd numbers of quarter-waves, and showing what value of $Z_0$ will match a source impedance $Z_s$ at quarter-wave frequency, given load impedance $Z_L$. What apparently has not been available in the textbooks is a useful alternate picture of Fig. 1 that helps everyone to remember the essential effects.

![Figure 1](image)

Figure 1. Transmission line $Z_0$ of length $l$ and propagation constant $\gamma$ terminated by load impedance $Z_L$; input impedance $Z_{in}$ is desired. $Z_0$ and $Z_L$ can be complex impedances $Z_L(s)$ and $Z_0(s)$.

We now use Eq. (1) to derive a useful equivalent circuit for $Z_{in}$ that will illustrate, at a glance, some of the memorable properties of a terminated transmission line.
Equation (1), after dividing top and bottom by \( \sinh(\gamma l) \), can be written as a sum of two terms,

\[
Z_{in} = \frac{Z_{L} Z_{0} \coth(\gamma l)}{Z_{0} \coth(\gamma l) + Z_{L}} + \frac{Z_{0}^{2}}{Z_{0} \coth(\gamma l) + Z_{L}}. \quad (2)
\]

The first term can already be recognized as the parallel impedance of \( Z_{L} \) and \( Z_{0} \coth(\gamma l) \), to which we will return shortly. Now multiply the second term of (2) by “1” in the following way:

\[
Z_{in} = \frac{Z_{L} Z_{0} \coth(\gamma l)}{Z_{0} \coth(\gamma l) + Z_{L}} + \frac{Z_{0}^{2}}{Z_{0} \coth(\gamma l) + Z_{L}} \frac{\tanh(\gamma l)}{\tanh(\gamma l)} \frac{Z_{0}}{Z_{0} \frac{Z_{L}}{Z_{L}}}. \quad (3)
\]

The various new factors in the second term can be associated with top and bottom and multiplied out to give

\[
Z_{in} = \frac{Z_{L} Z_{0} \coth(\gamma l)}{Z_{0} \coth(\gamma l) + Z_{L}} + \left[ \frac{Z_{0}^{2}}{Z_{L}} \right] Z_{0} \tanh(\gamma l) + \frac{Z_{0}^{2}}{Z_{0} \tanh(\gamma l) + Z_{L} \tanh(\gamma l)} \frac{Z_{0}}{Z_{0} \frac{Z_{L}}{Z_{L}}}. \quad (4)
\]

Now both terms of \( Z_{in} \) are parallel impedances, and they add in series. This means that

\[
Z_{in} = \left[ \frac{Z_{0}^{2}}{Z_{L}} \right] Z_{0} \tanh(\gamma l) + Z_{L} \frac{Z_{0} \coth(\gamma l)}{Z_{0} \coth(\gamma l) + Z_{L}}. \quad (5)
\]

The \( Z_{0} \coth \) and \( Z_{0} \tanh \) factors are well known as open and shorted stub impedances, respectively, so the new equivalent circuit is as shown in Figure 2.

**Figure 2.** New equivalent circuit of Fig. 1 \( Z_{in} \) using only lumped impedances and open and shorted stubs.

Now the important features of the arbitrarily loaded transmission line are clear from Fig. 2, as long as we remember what open and shorted stubs do. Recall that at quarter-wave frequency (also any odd number of quarter-waves), an open stub becomes a short, and a shorted stub becomes open. This tells
us immediately that at an odd number of quarter-waves, \( Z_L \) disappears and the “inversion impedance” \( Z_0^2 / Z_L \) emerges. Also, at zero frequency or at an even number of quarter waves, \( Z_L \) is restored and the inversion impedance disappears. The circuit is also correct for intermediate frequencies and for transients, and gives one a feel for what happens off-resonance, or in time domain, without using a computer. Finally, the well-known matching impedance problem is readily solved, i.e., for \( Z_m \) to match to a source impedance \( Z_s \), given \( Z_0 \) choose a quarter wavelength line section (resulting in the inversion impedance) and therefore choose \( Z_0 = \sqrt{Z_s Z_L} \).

This picture of the impedance \( Z_m \) of an arbitrarily terminated transmission line does not ordinarily appear in the related engineering textbooks, yet it lucidly illustrates the major features of unmatched line impedance that we all strive to recall and use. I hope that readers will find it useful.

References:

TIMOTHY J. MALONEY

timothy.j.maloney@intel.com

Timothy J. Maloney received an S.B. degree in physics from the Massachusetts Institute of Technology in 1971, an M.S. in physics from Cornell University in 1973, and a Ph.D. in electrical engineering from Cornell in 1976, where he was a National Science Foundation Fellow. He was a Postdoctoral Associate at Cornell until 1977, when he joined the Central Research Laboratory of Varian Associates, Palo Alto, CA. At Varian until 1984, he worked on III-V semiconductor photocathodes, solar cells and microwave devices, as well as silicon molecular beam epitaxy and MOS process technology. Since 1984 he has been with Intel Corp., Santa Clara, CA, where he has been concerned with integrated circuit ESD protection, CMOS latchup testing, fab process reliability, signal integrity, system ESD testing, and design and testing of standard IC layouts. He is now a Senior Principal Engineer at Intel. He has received the Intel Achievement Award for his patented ESD protection devices, which have achieved breakthrough ESD performance enhancements for a wide variety of Intel products. He now holds thirty-three patents, with several more pending.

Dr. Maloney received Best Paper Awards for his contributions to the EOS/ESD Symposium in 1986 and 1990, was General Chairman for the 1992 EOS/ESD Symposium, and received the ESD Association’s Outstanding Contributions Award in 1995. He has taught short courses at UCLA, University of Wisconsin, and UC Berkeley. He is co-author of a book, "Basic ESD and I/O Design" (Wiley, 1998), and is a Fellow of the IEEE.